

BACKSTEPPING DESIGN FOR INCREMENTAL STABILITY

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ABSTRACT. Stability is arguably one of the core concepts upon which our understanding of dynamical and control systems has been built. The related notion of incremental stability, however, has received much less attention until recently, when it was successfully used as a tool for the analysis and design of intrinsic observers, output regulation of nonlinear systems, frequency estimators, synchronization of coupled identical dynamical systems, symbolic models for nonlinear control systems, and bio-molecular systems. However, most of the existing controller design techniques provide controllers enforcing stability rather than incremental stability. Hence, there is a growing need to extend existing methods or develop new ones for the purpose of designing incrementally stabilizing controllers. In this paper, we develop a backstepping design approach for incremental stability. The effectiveness of the proposed method is illustrated by synthesizing a controller rendering a synchronous generator incrementally stable.

1. INTRODUCTION

Stability is a property of dynamical systems comparing trajectories with an equilibrium point or with a particular trajectory. Incremental stability is a stronger property comparing arbitrary trajectories with themselves, rather than with an equilibrium point or with a particular trajectory. It is well-known that for linear systems incremental stability is equivalent to stability. For nonlinear systems, incremental stability is a stronger property requiring separate concepts and techniques for its study.

The notion of incremental stability has a long history that can be traced back to the work of Zames in the 60's, as described in [Zam96]. In [Zam63], incremental stability is introduced and studied under the input-output setting where control systems are regarded as operators mapping input signals to output signals. Incremental stability then arises naturally by considering the Lipschitz constant of the operator. A modern treatment of incremental stability, based on Lyapunov methods, appeared only recently in [Ang02] where incremental global asymptotic stability (δ -GAS) and incremental input-to-state stability (δ -ISS) were defined in a state-space setting. The notion of δ -GAS was defined by requiring the Euclidean distance between two arbitrary system trajectories to converge¹ to zero. Lyapunov characterizations of δ -GAS and δ -ISS were also given in [Ang02]. There are two other stability properties related to incremental stability that have an equally long, if not longer, history.

The first is the notion of convergent system which, according to [PPvdWN04], was introduced in the 60's by B. P. Demidovich in [Dem61, Dem67]. A system is convergent if all the system trajectories converge¹ to a bounded trajectory. Furthermore, Demidovich also introduced a sufficient condition for a system to be convergent, called the Demidovich's condition in [PvdWN05]. Since incremental stability requires every trajectory to converge to every other trajectory, an incrementally stable system is also a convergent system whenever a bounded trajectory exists.

The second stability property is contractivity and was introduced in the control community by Lohmiller and Slotine in [LS98] although it had been studied before in the mathematical community [Jou05]. Rather than comparing trajectories, the notion of contracting system is infinitesimal and requires the decrease of a suitable quantity, defined through a Riemannian metric, along trajectories. The definition of contracting system can

This work has been partially supported by the National Science Foundation award 0717188 and 0820061.

¹A suitable "small overshoot" requirement is also included in the definition.

be seen as a generalization of the Demidovich's condition. Note that while the Demidovich's condition was introduced as a sufficient condition for a system to be convergent, its generalization in [LS98] was directly used as the definition of contracting system.

Although both contractivity as well as convergence are coordinate independent properties, this is no longer the case with incremental stability. In this paper, however, we work with a variation of incremental stability that is coordinate invariant. This is achieved by no longer insisting on the distance between trajectories being measured by the Euclidean metric.

The number of applications of incremental stability has increased in the past years. Examples include intrinsic observer design [AR03], consensus problems in complex networks [WS05], output regulation of nonlinear systems [PvdWN05], design of frequency estimators [SK08b], synchronization of coupled identical dynamical systems [RdBS09], construction of symbolic models for nonlinear control systems [PGT08, PT09, GPT09], and the analysis of bio-molecular systems [RdB09]. Our motivation comes from symbolic control where incremental stability was identified as a key property enabling the construction of finite abstractions of nonlinear control systems [PGT08, PT09, GPT09, ZPJT10]. Hence, there is a growing need for design methods providing controllers enforcing incremental stability since most of the existing design methods guarantee stability rather than incremental stability.

Related work includes the recent design results enforcing the convergent system property through the solution of linear matrix inequalities (LMIs) [PvdWN05, PvdWN07, vdWP08]. In contrast, the results presented in this paper do not require the solution of LMIs and existence of controllers is always guaranteed. Backstepping design methods for incremental global asymptotic stability² for parametric-strict-feedback³ systems were proposed before in [JL02, SK08a], and [SK09]. In this paper, we generalize the results in [JL02, SK08a], and [SK09] by:

- 1) developing a backstepping design method providing controllers enforcing incremental input-to-state stability⁴ and not simply incremental global asymptotic stability;
- 2) enlarging the class of control systems from parametric-strict-feedback to strict-feedback⁵ form.

The proposed approach was inspired by the original backstepping method described, for example, in [KKK95]. Like the original backstepping method, which provides a recursive way of constructing controllers as well as Lyapunov functions, the approach proposed in this paper provides a recursive way of constructing controllers as well as contraction metrics. Our design approach is illustrated by designing a controller rendering a synchronous generator incrementally stable.

2. CONTROL SYSTEMS AND STABILITY NOTIONS

2.1. Notation. The symbols \mathbb{R} , \mathbb{R}^+ and \mathbb{R}_0^+ denote the set of real, positive, and nonnegative real numbers, respectively. The symbols I_m , and 0_m denote the identity and zero matrices in $\mathbb{R}^{m \times m}$. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the i -th element of x , and by $\|x\|$ the Euclidean norm of x ; we recall that $\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$. Given a measurable function $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty := (\text{ess})\sup\{\|f(t)\|, t \geq 0\}$. A continuous function $\gamma : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, is said to belong to class \mathcal{K} if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class \mathcal{K}_∞ if $\gamma \in \mathcal{K}$ and $\gamma(r) \rightarrow \infty$ as $r \rightarrow \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is said to belong to class \mathcal{KL} if, for each fixed s , the map $\beta(r, s)$ belongs to class \mathcal{K}_∞ with respect to r and, for each fixed nonzero r , the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$. If $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth function with a smooth inverse, called a diffeomorphism, and if $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth map, we denote by ϕ_*X the map defined by $(\phi_*X)(y) = \frac{\partial \phi}{\partial x}|_{x=\phi^{-1}(y)} X \circ \phi^{-1}(y)$. Let now $G : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ be a smooth map. The

²Understood in the sense of Definition 2.2.

³See equation (3.1) or [KKK95] for a definition.

⁴Understood in the sense of Definition 2.3.

⁵See equation (3.18) or [KKK95] for a definition.

notation $\phi^*G : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ denotes the smooth map $(\phi^*G)(x) = (\frac{\partial \phi}{\partial x})^T G(\phi(x)) (\frac{\partial \phi}{\partial x})$. A Riemannian metric $G : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ is a smooth map on \mathbb{R}^n such that, for any $x \in \mathbb{R}^n$, $G(x)$ is a symmetric positive definite matrix [Lee03]. For any $x \in \mathbb{R}^n$ and smooth functions $I, J : \mathbb{R}^n \rightarrow \mathbb{R}$, one can define the scalar function $\langle I, J \rangle_G$ as $I^T(x)G(x)J(x)$. We will still use the notation $\langle I, J \rangle_G$ to denote $I^T G J$ even if G does not represent any Riemannian metric. A function $\mathbf{d} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ is a metric on \mathbb{R}^n if for any $x, y, z \in \mathbb{R}^n$, the following three conditions are satisfied: i) $\mathbf{d}(x, y) = 0$ if and only if $x = y$; ii) $\mathbf{d}(x, y) = \mathbf{d}(y, x)$; and iii) $\mathbf{d}(x, z) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z)$. We use the pair $(\mathbb{R}^n, \mathbf{d})$ to denote a metric space \mathbb{R}^n equipped with the metric \mathbf{d} . We use the notation \mathbf{d}_G to denote the Riemannian distance function provided by the Riemannian metric G [Lee03].

2.2. Control Systems. The class of control systems that we consider in this paper is formalized in the following definition.

Definition 2.1. *A control system is a quadruple:*

$$\Sigma = (\mathbb{R}^n, \mathbf{U}, \mathcal{U}, f),$$

where:

- \mathbb{R}^n is the state space;
- $\mathbf{U} \subseteq \mathbb{R}^m$ is the input set which is compact and convex;
- \mathcal{U} is the set of all measurable functions of time from intervals of the form $]a, b[\subseteq \mathbb{R}$ to \mathbf{U} with $a < 0$ and $b > 0$;
- $f : \mathbb{R}^n \times \mathbf{U} \rightarrow \mathbb{R}^n$ is a continuous map satisfying the following Lipschitz assumption: for every compact set $Q \subset \mathbb{R}^n$, there exists a constant $Z \in \mathbb{R}^+$ such that $\|f(x, u) - f(y, u)\| \leq Z\|x - y\|$ for all $x, y \in Q$ and all $u \in \mathbf{U}$.

A curve $\xi :]a, b[\rightarrow \mathbb{R}^n$ is said to be a *trajectory* of Σ if there exists $v \in \mathcal{U}$ satisfying:

$$(2.1) \quad \dot{\xi}(t) = f(\xi(t), v(t)),$$

for almost all $t \in]a, b[$. We also write $\xi_{xv}(t)$ to denote the point reached at time t under the input v from initial condition $x = \xi_{xv}(0)$; this point is uniquely determined, since the assumptions on f ensure existence and uniqueness of trajectories [Son98]. A control system Σ is said to be forward complete if every trajectory is defined on an interval of the form $]a, \infty[$. Sufficient and necessary conditions for a system to be forward complete can be found in [AS99]. A control system Σ is said to be smooth if f is an infinitely differentiable function of its arguments.

2.3. Stability notions. We start by introducing the following definitions which were inspired by the notions of incremental global asymptotic stability (δ -GAS) and incremental input-to-state stability (δ -ISS) presented in [Ang02].

Definition 2.2. *A control system Σ is incrementally globally asymptotically stable (δ_\exists -GAS) if it is forward complete and there exist a metric \mathbf{d} and a \mathcal{KL} function β such that for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$ and any $v \in \mathcal{U}$ the following condition is satisfied:*

$$(2.2) \quad \mathbf{d}(\xi_{xv}(t), \xi_{x'v}(t)) \leq \beta(\mathbf{d}(x, x'), t).$$

While δ -GAS, as defined in [Ang02], requires the metric \mathbf{d} to be the Euclidean metric, Definition 2.2 only requires the existence of a metric; hence, the existential quantifier in the acronym δ_\exists -GAS. We note that while δ -GAS is not invariant under changes of coordinates, δ_\exists -GAS is. If $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a bijective change of coordinates, inequality (2.2) transforms under ϕ to:

$$\mathbf{d}'(\phi \circ \xi_{xv}(t), \phi \circ \xi_{x'v}(t)) \leq \beta(\mathbf{d}'(\phi(x), \phi(x')), t),$$

where $\mathbf{d}'(y, y') = \mathbf{d}(\phi^{-1}(y), \phi^{-1}(y'))$. We shall return to the comparison between δ_\exists -GAS and δ -GAS in Subsection 2.4. Nevertheless, when the origin is an equilibrium point for Σ and the map $\psi : \mathbb{R}^n \rightarrow \mathbb{R}_0^+$, defined

by $\psi(x) = \mathbf{d}(x, 0)$, is continuous and radially unbounded⁶, both δ_{\exists} -GAS and δ -GAS imply global asymptotic stability.

Definition 2.3. A control system Σ is incrementally input-to-state stable (δ_{\exists} -ISS) if it is forward complete and there exist a metric \mathbf{d} , a \mathcal{KL} function β , and a \mathcal{K}_{∞} function γ such that for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$, and any $v, v' \in \mathcal{U}$ the following condition is satisfied:

$$(2.3) \quad \mathbf{d}(\xi_{xv}(t), \xi_{x'v'}(t)) \leq \beta(\mathbf{d}(x, x'), t) + \gamma(\|v - v'\|_{\infty}).$$

By observing (2.2) and (2.3), it is readily seen that δ_{\exists} -ISS implies δ_{\exists} -GAS while the converse is not true in general. Moreover, whenever the metric \mathbf{d} is the Euclidean metric, δ_{\exists} -ISS becomes δ -ISS as defined in [Ang02]. We note that while δ -ISS is not invariant under changes of coordinates, δ_{\exists} -ISS is. Once again, although δ_{\exists} -ISS is not equivalent to δ -ISS, both notions imply input-to-state stability whenever the origin is an equilibrium point for Σ and the map $\psi : \mathbb{R}^n \rightarrow \mathbb{R}_0^+$, defined by $\psi(x) = \mathbf{d}(x, 0)$, is continuous and radially unbounded.

2.4. Descriptions of incremental stability. One of the methods for checking incremental stability properties consists in using Lyapunov functions. The Lyapunov characterizations of δ -GAS and δ -ISS were developed in [Ang02]. In this paper we follow an alternative approach based on contraction metrics describing δ_{\exists} -GAS and δ_{\exists} -ISS properties, when \mathbf{d} is the Riemannian distance function. The notion of contraction metric was popularized in control theory by the work of Lohmiller and Slotine [LS98] that relies on variational systems.

The variational system associated with a smooth control system $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$, when we only have state variations, is given by the differential equation:

$$(2.4) \quad \frac{d}{dt}(\delta\xi) = \frac{\partial f}{\partial x} \Big|_{\substack{x=\xi \\ u=v}} \delta\xi,$$

for any $v \in \mathcal{U}$ and where $\delta\xi$ is the variation⁷ of a trajectory ξ of Σ . More details about the variational system can be found in [CS87]. Similarly, the variational system associated with a smooth control system $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$, when we have state and input variations, is given by the differential equation:

$$(2.5) \quad \frac{d}{dt}(\delta\xi) = \frac{\partial f}{\partial x} \Big|_{\substack{x=\xi \\ u=v}} \delta\xi + \frac{\partial f}{\partial u} \Big|_{\substack{x=\xi \\ u=v}} \delta v,$$

where $\delta\xi$ and δv are variations of a state trajectory ξ and an input curve v of Σ , respectively.

The following definition is adapted from [LS98]:

Definition 2.4. Let $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ be a smooth control system equipped with a Riemannian metric G . The metric G is said to be a contraction metric, with respect to states, for system Σ if there exists some $\lambda \in \mathbb{R}^+$ such that:

$$(2.6) \quad \langle X, X \rangle_F \leq -\lambda \langle X, X \rangle_G$$

for $F(x, u) = \left(\frac{\partial f}{\partial x} \right)^T G(x) + G(x) \frac{\partial f}{\partial x} + \frac{\partial G}{\partial x} f(x, u)$, any $u \in \mathcal{U}$, $X \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, or equivalently:

$$(2.7) \quad X^T \left(\left(\frac{\partial f}{\partial x} \right)^T G(x) + G(x) \frac{\partial f}{\partial x} + \frac{\partial G}{\partial x} f(x, u) \right) X \leq -\lambda X^T G(x) X,$$

where the constant λ is called contraction rate.

When the metric G is constant, the condition (2.6) or (2.7) is known as the Demidovich's condition [PvdWN05]. It is shown in [PvdWN05] that such condition implies incremental stability and the convergent system property.

⁶Under the stated assumptions it can be shown that $\underline{\alpha}(\|x\|) \leq \psi(x) \leq \overline{\alpha}(\|x\|)$ for \mathcal{K}_{∞} functions $\underline{\alpha}$ and $\overline{\alpha}$.

⁷The variation $\delta\xi$ can be formally defined by considering a family of trajectories $\xi_{xv}(t, \epsilon)$ parametrized by $\epsilon \in \mathbb{R}$. The variation of the state is then $\delta\xi = \frac{\partial \xi_{xv}}{\partial \epsilon}$.

Note that the inequality (2.6) or (2.7) implies:

$$(2.8) \quad \frac{d}{dt} \langle \delta \xi, \delta \xi \rangle_G \leq -\lambda \langle \delta \xi, \delta \xi \rangle_G,$$

when we only have state variations and $\delta \xi$ is the variation of a state trajectory ξ of Σ .

The following theorem shows that the inequality (2.6) implies δ_\exists -GAS.

Theorem 2.5. *Let $\Sigma = (\mathbb{R}^n, \mathcal{U}, \mathcal{U}, f)$ be a smooth control system equipped with a Riemannian metric G . If G is a contraction metric, with respect to states, for Σ and if $(\mathbb{R}^n, \mathbf{d}_G)$ is a complete metric space⁸, then Σ is δ_\exists -GAS.*

Different variations of this result appeared in [LS98] and [AR03]; see [AR03] for a concise proof and [ZPJT10] for a more detailed treatment including the completeness assumption. It is also shown in [ZPJT10] that the additional assumption $\underline{\omega} \langle X, X \rangle_{I_n} \leq \langle X, X \rangle_G \leq \overline{\omega} \langle X, X \rangle_{I_n}$ for $\underline{\omega}, \overline{\omega} \in \mathbb{R}^+$ leads to the stronger conclusion that Σ is in fact δ -GAS.

If, in addition to state variations, we also allow for input variations we obtain the notion of contraction metric with respect to states and inputs.

Definition 2.6. *Let $\Sigma = (\mathbb{R}^n, \mathcal{U}, \mathcal{U}, f)$ be a smooth control system on \mathbb{R}^n equipped with a Riemannian metric G . The metric G is said to be a contraction metric, with respect to states and inputs, for system Σ if there exists some $\lambda \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}_0^+$ such that:*

$$(2.9) \quad \langle X, X \rangle_F + 2 \left\langle \frac{\partial f}{\partial u} Y, X \right\rangle_G \leq -\lambda \langle X, X \rangle_G + \alpha \langle X, X \rangle_G^{\frac{1}{2}} \langle Y, Y \rangle_{I_m}^{\frac{1}{2}}$$

for $F(x, u) = \left(\frac{\partial f}{\partial x} \right)^T G(x) + G(x) \frac{\partial f}{\partial x} + \frac{\partial G}{\partial x} f(x, u)$, any $X \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $u \in \mathcal{U}$, and $Y \in \mathbb{R}^m$, or equivalently:

$$(2.10) \quad \begin{aligned} & X^T \left(\left(\frac{\partial f}{\partial x} \right)^T G(x) + G(x) \frac{\partial f}{\partial x} + \frac{\partial G}{\partial x} f(x, u) \right) X + 2Y^T \left(\frac{\partial f}{\partial u} \right)^T G(x) X \\ & \leq -\lambda X^T G(x) X + \alpha (X^T G(x) X)^{\frac{1}{2}} (Y^T Y)^{\frac{1}{2}}, \end{aligned}$$

where the constant λ is called contraction rate.

Note that the inequality (2.9) or (2.10) implies:

$$(2.11) \quad \frac{d}{dt} \langle \delta \xi, \delta \xi \rangle_G \leq -\lambda \langle \delta \xi, \delta \xi \rangle_G + \alpha \langle \delta \xi, \delta \xi \rangle_G^{\frac{1}{2}} \langle \delta v, \delta v \rangle_{I_m}^{\frac{1}{2}},$$

when we have state and input variations and where $\delta \xi$ and δv are variations of a state trajectory ξ and an input curve v of Σ .

The following theorem shows that the inequality (2.9) implies δ_\exists -ISS.

Theorem 2.7 ([ZPJT10]). *Let $\Sigma = (\mathbb{R}^n, \mathcal{U}, \mathcal{U}, f)$ be a smooth control system on \mathbb{R}^n equipped with a Riemannian metric G . If the metric G is a contraction metric, with respect to states and inputs, for system Σ and $(\mathbb{R}^n, \mathbf{d}_G)$ is a complete metric space, then Σ is δ_\exists -ISS.*

Similarly to contraction metrics with respect to states, it shown in [ZPJT10] that Σ is δ -ISS when the inequalities $\underline{\omega} \langle X, X \rangle_{I_n} \leq \langle X, X \rangle_G \leq \overline{\omega} \langle X, X \rangle_{I_n}$ are satisfied for $\underline{\omega}, \overline{\omega} \in \mathbb{R}^+$.

In the next section, we propose a backstepping design procedure to render control systems incrementally stable.

⁸A metric space $(\mathbb{R}^n, \mathbf{d})$ is said to be complete if every Cauchy sequence of points in \mathbb{R}^n has a limit that is also in \mathbb{R}^n .

3. BACKSTEPPING DESIGN PROCEDURE

The method described here was inspired by the original backstepping described, for example, in [KKK95]. Consider the class of control systems $\Sigma = (\mathbb{R}^n, \mathcal{U}, \mathcal{U}, f)$ with f of the parametric-strict-feedback form [KKK95]:

$$(3.1) \quad \begin{aligned} f_1(x, u) &= h_1(x_1) + b_1 x_2, \\ f_2(x, u) &= h_2(x_1, x_2) + b_2 x_3, \\ &\vdots \\ f_{n-1}(x, u) &= h_{n-1}(x_1, \dots, x_{n-1}) + b_{n-1} x_n, \\ f_n(x, u) &= h_n(x) + g(x)u, \end{aligned}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathcal{U} \subset \mathbb{R}$ is the control input. The functions $h_i : \mathbb{R}^i \rightarrow \mathbb{R}$, for $i = 1, \dots, n$, and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ are smooth, $g(x) \neq 0$ over the domain of interest, and $b_i \in \mathbb{R}$, for $i = 1, \dots, n$, are nonzero constants.

We can now state one of the main results, describing a backstepping controller for control system (3.1).

Theorem 3.1. *For any control system $\Sigma = (\mathbb{R}^n, \mathcal{U}, \mathcal{U}, f)$ with f of the form (3.1) and for any $\lambda \in \mathbb{R}^+$, the state feedback control law:*

$$(3.2) \quad k(x, \hat{u}) = \frac{1}{g(x)} \left[k_n(x) - h_n(x) \right] + \frac{1}{g(x)} \hat{u},$$

where

$$\begin{aligned} k_l(x, \hat{u}) &= -b_{l-1} (x_{l-1} - \phi_{l-2}(x)) - \frac{\lambda}{2} (x_l - \phi_{l-1}(x)) + \frac{\partial \phi_{l-1}}{\partial x} f(x, k(x, \hat{u})), \text{ for } l = 1, \dots, n, \\ \phi_l(x) &= \frac{1}{b_l} \left[k_l(x) - h_l(x) \right], \text{ for } l = 1, \dots, n-1, \\ \phi_{-1}(x) &= \phi_0(x) = 0 \quad \forall x \in \mathbb{R}^n, \quad b_0 = 0, \text{ and } x_0 = 0, \end{aligned}$$

renders the control system Σ δ_3 -GAS.

Proof. Consider the following system:

$$(3.3) \quad \Sigma_l : \begin{cases} \dot{\eta}_l = F_l(\eta_l) + B_l \xi_l, \\ \dot{\xi}_l = k_l(\eta_l, \xi_l), \end{cases}$$

where $\eta_l = [\xi_1, \dots, \xi_{l-1}]^T$, $B_l = [0, \dots, 0, b_{l-1}]^T \in \mathbb{R}^{l-1}$, $z_l = [y_l^T \ x_l]^T \in \mathbb{R}^l$ is the state of Σ_l , $y_l = [x_1, \dots, x_{l-1}]^T$, and $F_l(y_l) = [f_1(x, u), \dots, f_{l-2}(x, u), h_{l-1}(x_1, \dots, x_{l-1})]^T$. By using induction on l , we show that the metric G_l , defined by:

$$(3.4) \quad G_l(y_l) = \begin{bmatrix} G_{l-1}(y_{l-1}) + \left(\frac{\partial \phi_{l-1}}{\partial y_l} \right)^T \frac{\partial \phi_{l-1}}{\partial y_l} & - \left(\frac{\partial \phi_{l-1}}{\partial y_l} \right)^T \\ - \frac{\partial \phi_{l-1}}{\partial y_l} & 1 \end{bmatrix},$$

is a contraction metric, with respect to states, for the system (3.3) with contraction rate λ . For $l = 1$, it can be easily checked that $G_1(y_1) = 1$ is a contraction metric, with respect to states, with the contraction rate λ for the scalar system:

$$\Sigma_1 : \dot{\xi}_1 = k_1(\xi_1) = -\frac{\lambda}{2} \xi_1.$$

Assume that the metric G_{k-1} is a contraction metric, with respect to states, for the system Σ_{k-1} , for some $2 \leq k \leq n$, and with contraction rate λ . This implies:

$$(3.5) \quad \begin{aligned} \begin{bmatrix} Y^T & X \end{bmatrix} \left(\left(\frac{\partial(F_k + B_k \phi_{k-1})}{\partial y_k} \right)^T G_{k-1}(y_{k-1}) + G_{k-1}(y_{k-1}) \frac{\partial(F_k + B_k \phi_{k-1})}{\partial y_k} + \frac{\partial G_{k-1}}{\partial y_k} (F_k + B_k \phi_{k-1}) \right) \begin{bmatrix} Y \\ X \end{bmatrix} \\ \leq -\lambda \begin{bmatrix} Y^T & X \end{bmatrix} G_{k-1}(y_{k-1}) \begin{bmatrix} Y \\ X \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
(3.9) \quad [Y^T \ X] & \left(\left(\frac{\partial [F_k^T + B_k^T x_k \ k_k(x, \hat{u})]^T}{\partial z_k} \right)^T G_k(y_k) + G_k(y_k) \frac{\partial [F_k^T + B_k^T x_k \ k_k(x, \hat{u})]^T}{\partial z_k} + \dot{G}_k(y_k) \right) \begin{bmatrix} Y \\ X \end{bmatrix} = \\
& [Y^T \ X] \left(\begin{bmatrix} \frac{\partial F_k}{\partial y_k} & B_k \\ (F_k + B_k x_k)^T \frac{\partial^2 \phi_{k-1}}{\partial y_k^2} + \frac{\partial \phi_{k-1}}{\partial y_k} \frac{\partial F_k}{\partial y_k} + \frac{\lambda}{2} \frac{\partial \phi_{k-1}}{\partial y_k} - B_k^T G_{k-1}(y_{k-1}) & -\frac{\lambda}{2} + \frac{\partial \phi_{k-1}}{\partial y_k} B_k \end{bmatrix}^T \right. \\
& \left. \begin{bmatrix} G_{k-1}(y_{k-1}) + \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \frac{\partial \phi_{k-1}}{\partial y_k} & - \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \\ -\frac{\partial \phi_{k-1}}{\partial y_k} & 1 \end{bmatrix} + \begin{bmatrix} G_{k-1}(y_{k-1}) + \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \frac{\partial \phi_{k-1}}{\partial y_k} & - \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \\ -\frac{\partial \phi_{k-1}}{\partial y_k} & 1 \end{bmatrix} \right. \\
& \left. \begin{bmatrix} \frac{\partial F_k}{\partial y_k} & B_k \\ (F_k + B_k x_k)^T \frac{\partial^2 \phi_{k-1}}{\partial y_k^2} + \frac{\partial \phi_{k-1}}{\partial y_k} \frac{\partial F_k}{\partial y_k} + \frac{\lambda}{2} \frac{\partial \phi_{k-1}}{\partial y_k} - B_k^T G_{k-1}(y_{k-1}) & -\frac{\lambda}{2} + \frac{\partial \phi_{k-1}}{\partial y_k} B_k \end{bmatrix} + \right. \\
& \left. \begin{bmatrix} \frac{\partial G_{k-1}}{\partial y_k} (F_k + B_k x_k) + \frac{\partial^2 \phi_{k-1}}{\partial y_k^2} (F_k + B_k x_k) \frac{\partial \phi_{k-1}}{\partial y_k} + \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T (F_k + B_k x_k)^T \frac{\partial^2 \phi_{k-1}}{\partial y_k^2} & -\frac{\partial^2 \phi_{k-1}}{\partial y_k^2} (F_k + B_k x_k) \\ - (F_k + B_k x_k)^T \frac{\partial^2 \phi_{k-1}}{\partial y_k^2} & 0 \end{bmatrix} \right) \\
& \begin{bmatrix} Y \\ X \end{bmatrix} = [Y^T \ X] \cdot \\
& \left[\begin{bmatrix} \left(\frac{\partial (F_k + B_k \phi_{k-1})}{\partial y_k} \right)^T G_{k-1}(y_{k-1}) + G_{k-1}(y_{k-1}) \frac{\partial (F_k + B_k \phi_{k-1})}{\partial y_k} + \frac{\partial G_{k-1}}{\partial y_k} F_k & -\lambda \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \frac{\partial \phi_{k-1}}{\partial y_k} & \lambda \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \\ \lambda \frac{\partial \phi_{k-1}}{\partial y_k} & & -\lambda \end{bmatrix} \right] \\
& \begin{bmatrix} Y \\ X \end{bmatrix} \leq -\lambda [Y^T \ X] G_k(y_k) \begin{bmatrix} Y \\ X \end{bmatrix}.
\end{aligned}$$

for any $Y \in \mathbb{R}^{k-2}$, and $X \in \mathbb{R}$. Since the metric G_{k-1} is only a function of $y_{k-1} = [x_1, \dots, x_{k-2}]^T$, and the vector B_k has zero entries except for the last entry, it can be easily shown that $\frac{\partial G_{k-1}}{\partial y_k} B_k = 0_{k-1}$, and the inequality (3.5) reduces to:

$$\begin{aligned}
(3.6) \quad [Y^T \ X] & \left(\left(\frac{\partial (F_k + B_k \phi_{k-1})}{\partial y_k} \right)^T G_{k-1}(y_{k-1}) + G_{k-1}(y_{k-1}) \frac{\partial (F_k + B_k \phi_{k-1})}{\partial y_k} + \frac{\partial G_{k-1}}{\partial y_k} F_k \right) \begin{bmatrix} Y \\ X \end{bmatrix} \\
& \leq -\lambda [Y^T \ X] G_{k-1}(y_{k-1}) \begin{bmatrix} Y \\ X \end{bmatrix}.
\end{aligned}$$

Now, we show that:

$$(3.7) \quad G_k(y_k) = \begin{bmatrix} G_{k-1}(y_{k-1}) + \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \frac{\partial \phi_{k-1}}{\partial y_k} & - \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \\ -\frac{\partial \phi_{k-1}}{\partial y_k} & 1 \end{bmatrix},$$

is a contraction metric, with respect to states, for the system Σ_k . For any nonzero vector $[Y^T \ X]^T \in \mathbb{R}^k$, we have:

$$\begin{aligned}
(3.8) \quad [Y^T \ X] G_k(y_k) \begin{bmatrix} Y \\ X \end{bmatrix} & = [Y^T \ X] \begin{bmatrix} G_{k-1}(y_{k-1}) + \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \frac{\partial \phi_{k-1}}{\partial y_k} & - \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \\ -\frac{\partial \phi_{k-1}}{\partial y_k} & 1 \end{bmatrix} \begin{bmatrix} Y \\ X \end{bmatrix} \\
& = Y^T G_{k-1}(y_{k-1}) Y + \left(\frac{\partial \phi_{k-1}}{\partial y_k} Y - X \right)^2.
\end{aligned}$$

If $Y \in \mathbb{R}^{k-1}$ is the zero vector, X must be nonzero implying that the equation (3.8) is equal to X^2 which is positive. On the other hand, if $Y \in \mathbb{R}^{k-1}$ is nonzero, $Y^T G_{k-1}(y_{k-1}) Y$ is a positive scalar because G_{k-1} is a Riemannian metric. Hence, G_k is positive definite. Using the inequality (3.6), the long algebraic manipulations in (3.9) show that G_k satisfies (2.7) with the contraction rate λ . Hence, the metric G_k is a contraction metric, with respect to states, for the system Σ_k . Therefore, for any $l \leq n$, the metric G_l is a contraction metric, with

respect to states, for the system (3.3) and with the contraction rate λ .

The proposed control law (3.2), transforms a control system of the form (3.1) into:

$$(3.10) \quad \Sigma_n : \begin{cases} \dot{\eta}_n = F_n(\eta_n) + B_n \xi_n, \\ \dot{\xi}_n = k_n(\eta_n, \xi_n) + \hat{v}. \end{cases}$$

It can be easily checked that \hat{v} does not appear in the variation of Σ_n when we only have state variations. Since the metric G_n is not a function of the n -th state, its derivative with respect to time does not include \hat{v} . Hence, we can apply the induction results to Σ_n to conclude that the metric G_n is a contraction metric, with respect to states, for Σ_n and with the contraction rate λ . Moreover, it can be readily seen that $G_n = \psi^* I_n$, where

$$(3.11) \quad \psi(x) = \begin{bmatrix} x_1 \\ x_2 - \phi_1(x) \\ x_3 - \phi_2(x) \\ \vdots \\ x_n - \phi_{n-1}(x) \end{bmatrix}.$$

Note that \mathbf{d}_{I_n} is just the Euclidean metric and we know that $(\mathbb{R}^n, \mathbf{d}_{I_n})$ is a complete metric space. Moreover, since $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isometry⁹, $(\mathbb{R}^n, \mathbf{d}_{G_n})$ is also a complete metric space [Lee03]. By using Theorem 2.5, we conclude that a control system of the form (3.1), equipped with the state feedback control law (3.2), is δ_{\exists} -GAS. The δ_{\exists} -GAS condition (2.2), as shown in [AR03], is given by:

$$(3.12) \quad \mathbf{d}_{G_n}(\xi_{xv}(t), \xi_{x'v}(t)) \leq e^{-\frac{\lambda}{2}t} \mathbf{d}_{G_n}(x, x').$$

□

Remark 3.2. The contraction metric $G_n(y_n)$, with respect to states, for the control system (3.1), equipped with the state feedback control law (3.2), is given by:

$$(3.13) \quad \begin{bmatrix} \left[\begin{bmatrix} 1 + \left(\frac{\partial \phi_1}{\partial y_2} \right)^T \frac{\partial \phi_1}{\partial y_2} & - \left(\frac{\partial \phi_1}{\partial y_2} \right)^T \\ - \frac{\partial \phi_1}{\partial y_2} & 1 \end{bmatrix} + \left(\frac{\partial \phi_2}{\partial y_3} \right)^T \frac{\partial \phi_2}{\partial y_3} & - \left(\frac{\partial \phi_2}{\partial y_3} \right)^T \\ \vdots & \vdots \\ - \frac{\partial \phi_{n-1}}{\partial y_n} & 1 \end{bmatrix} + \left(\frac{\partial \phi_{n-1}}{\partial y_n} \right)^T \frac{\partial \phi_{n-1}}{\partial y_n} & - \left(\frac{\partial \phi_{n-1}}{\partial y_n} \right)^T \right],$$

where $y_l = [x_1, \dots, x_{l-1}]^T$, for $l = 2, \dots, n$, and the contraction rate is λ .

Remark 3.3. It can be checked that the function

$$V(x) = \frac{1}{2} \sum_{l=0}^{n-1} (x_{l+1} - \phi_l(x))^2,$$

is a Lyapunov function [Kha96] for the control system (3.1), equipped with the state feedback control law (3.2) when $\hat{u} = 0$. Moreover, the Hessian of $V(x)$ is equal to the contraction metric G_n , with respect to states, defined in (3.13).

In the next theorem, we show that control law (3.2) also enforces δ_{\exists} -ISS.

Theorem 3.4. For any control system $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ with f of the form (3.1) and for any $\lambda \in \mathbb{R}^+$, the state feedback control law:

$$(3.14) \quad k(x, \hat{u}) = \frac{1}{g(x)} \left[k_n(x) - h_n(x) \right] + \frac{1}{g(x)} \hat{u},$$

⁹Suppose M and \widetilde{M} are Riemannian manifolds with Riemannian metrics G and \widetilde{G} , respectively. A smooth map $\psi : M \rightarrow \widetilde{M}$ is called an isometry if it is a diffeomorphism satisfying $G = \psi^* \widetilde{G}$.

where

$$\begin{aligned} k_l(x, \hat{u}) &= -b_{l-1}(x_{l-1} - \phi_{l-2}(x)) - \frac{\lambda}{2}(x_l - \phi_{l-1}(x)) + \frac{\partial \phi_{l-1}}{\partial x} f(x, k(x, \hat{u})), \text{ for } l = 1, \dots, n, \\ \phi_l(x) &= \frac{1}{b_l} \left[k_l(x) - h_l(x) \right], \text{ for } l = 1, \dots, n-1, \\ \phi_{-1}(x) &= \phi_0(x) = 0 \quad \forall x \in \mathbb{R}^n, \quad b_0 = 0, \text{ and } x_0 = 0, \end{aligned}$$

renders the control system Σ δ_{\exists} -ISS with respect to the input \hat{v} .

Proof. Consider the following system:

$$(3.15) \quad \Sigma_l : \quad \dot{\eta}_l = F_l(\eta_l) + B_l \phi_{l-1}(\eta_l), \text{ for } l = 2, \dots, n,$$

where $\eta_l = [\xi_1, \dots, \xi_{l-1}]^T$, $B_l = [0, \dots, 0, b_{l-1}]^T \in \mathbb{R}^{l-1}$, $y_l^T = [x_1, \dots, x_{l-1}]^T$ is the state of Σ_l , and $F_l(\eta_l) = [f_1(\xi, v), \dots, f_{l-2}(\xi, v), h_{l-1}(\xi_1, \dots, \xi_{l-1})]^T$. As proved in Theorem 3.1, the metric G_l in (3.4) is a contraction metric, with respect to states, for the system Σ_l and with the contraction rate λ . The proposed control law (3.14), transforms a control system of the form (3.1) into:

$$(3.16) \quad \Sigma : \begin{cases} \dot{\eta}_n = F_n(\eta_n) + B_n \xi_n, \\ \dot{\xi}_n = k_n(\eta_n, \xi_n) + \hat{v}. \end{cases}$$

Now, we show that:

$$(3.17) \quad G_n(y_n) = \begin{bmatrix} G_{n-1}(y_{n-1}) + \left(\frac{\partial \phi_{n-1}}{\partial y_n} \right)^T \frac{\partial \phi_{n-1}}{\partial y_n} & - \left(\frac{\partial \phi_{n-1}}{\partial y_n} \right)^T \\ - \frac{\partial \phi_{n-1}}{\partial y_n} & 1 \end{bmatrix},$$

is a contraction metric, with respect to states and inputs, for the control system (3.16). For $n = 1$, it can be easily checked that $G_1(y_1) = 1$ is a contraction metric, with respect to states and inputs, with the contraction rate λ , satisfying (2.10) with $\alpha = 2$ for the scalar control system:

$$\Sigma : \dot{\xi}_1 = k_1(\xi_1) + \hat{v} = -\frac{\lambda}{2} \xi_1 + \hat{v}.$$

As proved in Theorem 3.1, $G_n(y_n)$ is positive definite. Using the inequality (3.6) for $k = n$, long algebraic manipulations similar to those in (3.9) show that G_n satisfies (2.10) with the contraction rate λ and $\alpha = 2$. Hence, the metric G_n is a contraction metric, with respect to states and inputs, for the control system (3.16). As explained in the proof of Theorem 3.1, we know that $(\mathbb{R}^n, \mathbf{d}_{G_n})$ is a complete metric space. By using Theorem 2.7, we conclude that a control system of the form (3.1), equipped with the state feedback control law (3.14), is δ_{\exists} -ISS with respect to \hat{v} . The δ_{\exists} -ISS condition (2.3), as shown in [ZPJT10], is given by:

$$\begin{aligned} \mathbf{d}_{G_n}(\xi_{x\hat{v}}(t), \xi_{x'\hat{v}'}(t)) &\leq e^{-\frac{\lambda}{2}t} \mathbf{d}_{G_n}(x, x') + \frac{2}{\lambda} \left(1 - e^{-\frac{\lambda}{2}t} \right) \|\hat{v} - \hat{v}'\|_{\infty} \\ &\leq e^{-\frac{\lambda}{2}t} \mathbf{d}_{G_n}(x, x') + \frac{2}{\lambda} \|\hat{v} - \hat{v}'\|_{\infty}. \end{aligned}$$

□

Remark 3.5. The contraction metric, with respect to states and inputs, for a control system of the form (3.1), equipped with the state feedback control law (3.14), is given by (3.13).

Remark 3.6. It can be shown that the function

$$V(x) = \frac{1}{2} \sum_{l=0}^{n-1} (x_{l+1} - \phi_l(x))^2,$$

is an Input-to-State Stability Lyapunov function [Kha96] with respect to \hat{v} for a control system of the form (3.1), equipped with the state feedback control law (3.14).

Now, we extend the results in Theorems 3.1 and 3.4 to the class of control systems $\Sigma = (\mathbb{R}^n, \mathbb{R}, \mathcal{U}, f)$ with f of the strict-feedback form [KKK95]:

$$\begin{aligned}
 f_1(x, u) &= h_1(x_1) + g_1(x_1)x_2, \\
 f_2(x, u) &= h_2(x_1, x_2) + g_2(x_1, x_2)x_3, \\
 &\vdots \\
 f_{n-1}(x, u) &= h_{n-1}(x_1, \dots, x_{n-1}) + g_{n-1}(x_1, \dots, x_{n-1})x_n, \\
 f_n(x, u) &= h_n(x) + g_n(x)u,
 \end{aligned}
 \tag{3.18}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{U} \subset \mathbb{R}$ is the control input. The functions $h_i : \mathbb{R}^i \rightarrow \mathbb{R}$, and $g_i : \mathbb{R}^i \rightarrow \mathbb{R}$, for $i = 1, \dots, n$, are smooth, and $g_i(x_1, \dots, x_i) \neq 0$ over the domain of interest.

In order to extend Theorems 3.1, and 3.4 to control systems of the form (3.18), we need the following technical lemmas.

Lemma 3.7. *Let $\Sigma = (\mathbb{R}^n, \mathbb{U}, \mathcal{U}, f)$ be a control system and let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth map with a smooth inverse. If the metric G is a contraction metric, with respect to states, for $\Sigma' = (\mathbb{R}^n, \mathbb{U}, \mathcal{U}, \phi_* f)$ and with contraction rate $\lambda \in \mathbb{R}^+$, then the metric $\phi^* G$ is a contraction metric, with respect to states, for the system Σ and with the contraction rate λ .*

Proof. Since G is a contraction metric, with respect to states, for the system Σ' and with the contraction rate λ , using the inequality (2.8), we have:

$$\frac{d}{dt} \langle \delta\eta, \delta\eta \rangle_G \leq -\lambda \langle \delta\eta, \delta\eta \rangle_G,
 \tag{3.19}$$

where $\delta\eta$ is variation of the state trajectory of Σ' . Since G is a metric and $\Theta(x) = \frac{\partial \phi}{\partial x}(x)$ is an invertible matrix¹⁰, it is readily seen that $(\phi^* G)(x)$ is a positive definite matrix. We now show that the metric $\phi^* G$ is a contraction metric, with respect to states, for the system Σ . For the coordinate transformation $\eta = \phi(\xi)$, we have:

$$\delta\eta = \Theta(\xi) \delta\xi.
 \tag{3.20}$$

By taking the derivative of (3.20) with respect to time, we obtain:

$$\frac{d}{dt} \delta\eta = \dot{\Theta}(\xi) \delta\xi + \Theta(\xi) \frac{d}{dt} \delta\xi.
 \tag{3.21}$$

Using (3.19), (3.20), and (3.21), we obtain:

$$\begin{aligned}
 \frac{d}{dt} \langle \delta\eta, \delta\eta \rangle_G &= \left(\frac{d}{dt} \delta\eta \right)^T G \delta\eta + \delta\eta^T G \frac{d}{dt} \delta\eta + \delta\eta^T \dot{G} \delta\eta \\
 &= \left(\dot{\Theta} \delta\xi + \Theta \frac{d}{dt} \delta\xi \right)^T G \Theta \delta\xi + (\Theta \delta\xi)^T G \left(\dot{\Theta} \delta\xi + \Theta \frac{d}{dt} \delta\xi \right) + \delta\eta^T \dot{G} \delta\eta \\
 &= \left(\frac{d}{dt} \delta\xi \right)^T \phi^* G \delta\xi + \delta\xi^T \phi^* G \frac{d}{dt} \delta\xi + \delta\xi \frac{d}{dt} (\phi^* G) \delta\xi = \frac{d}{dt} \langle \delta\xi, \delta\xi \rangle_{\phi^* G} \\
 &\leq -\lambda \langle \delta\eta, \delta\eta \rangle_G = -\lambda \delta\eta^T G \delta\eta = -\lambda (\Theta \delta\xi)^T G \Theta \delta\xi = -\lambda \delta\xi^T \phi^* G \delta\xi \\
 &= -\lambda \langle \delta\xi, \delta\xi \rangle_{\phi^* G}.
 \end{aligned}$$

Hence, the metric $\phi^* G$ is a contraction metric, with respect to states, for the system Σ and with the contraction rate λ . \square

In the next lemma, we extend the results of Lemma 3.7 to contraction with respect to states and inputs.

¹⁰For any smooth map $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with a smooth inverse, it is easy to show that $\frac{\partial \phi}{\partial x}(x)$ is an invertible matrix for any $x \in \mathbb{R}^n$.

Lemma 3.8. *Let $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ be a control system and let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth map with a smooth inverse. If the metric G is a contraction metric, with respect to states and inputs, satisfying (2.10) with contraction rate $\lambda \in \mathbb{R}^+$, and $\alpha \in \mathbb{R}_0^+$ for $\Sigma' = (\mathbb{R}^n, \mathcal{U}, \phi_* f)$, then the metric $\phi^* G$ is a contraction metric, with respect to states and inputs, satisfying (2.10) with the contraction rate λ , and the nonnegative constant α for the system Σ .*

Proof. Since G is a contraction metric, with respect to states and inputs, for the control system Σ' , satisfying (2.10) with the contraction rate λ , and $\alpha \in \mathbb{R}_0^+$, using the inequality (2.11), we have:

$$(3.22) \quad \frac{d}{dt} \langle \delta \eta, \delta \eta \rangle_G \leq -\lambda \langle \delta \eta, \delta \eta \rangle_G + \alpha \langle \delta \eta, \delta \eta \rangle_G^{\frac{1}{2}} \langle \delta v, \delta v \rangle_{I_m}^{\frac{1}{2}}.$$

Using (3.20), (3.21), (3.22), and the results of Lemma 3.7, we obtain:

$$\begin{aligned} \frac{d}{dt} \langle \delta \eta, \delta \eta \rangle_G &= \frac{d}{dt} \langle \delta \xi, \delta \xi \rangle_{\phi^* G} \\ &\leq -\lambda \langle \delta \eta, \delta \eta \rangle_G + \alpha \langle \delta \eta, \delta \eta \rangle_G^{\frac{1}{2}} \langle \delta v, \delta v \rangle_{I_m}^{\frac{1}{2}} \\ &= -\lambda \langle \delta \xi, \delta \xi \rangle_{\phi^* G} + \alpha \left((\Theta \delta \xi)^T G \Theta \delta \xi \right)^{\frac{1}{2}} \langle \delta v, \delta v \rangle_{I_m}^{\frac{1}{2}} \\ &= -\lambda \langle \delta \xi, \delta \xi \rangle_{\phi^* G} + \alpha \langle \delta \xi, \delta \xi \rangle_{\phi^* G}^{\frac{1}{2}} \langle \delta v, \delta v \rangle_{I_m}^{\frac{1}{2}}. \end{aligned}$$

Hence, the metric $\phi^* G$ is a contraction metric, with respect to states and inputs, satisfying (2.10) with the contraction rate λ , and the nonnegative constant α , for Σ . \square

We can now state the main result for a control system $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ with f of the form (3.18).

Theorem 3.9. *Let $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ be a control system where f is of the form (3.18). The state feedback control law $u = k(\phi(x), \hat{u})$, where k was defined in (3.2) and $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the smooth map (with smooth inverse) defined by:*

$$(3.23) \quad \phi(x) = \begin{bmatrix} x_1 \\ g_1(x_1)x_2 \\ g_1(x_1)g_2(x_1, x_2)x_3 \\ \vdots \\ \prod_{i=1}^{n-1} g_i(x_1, \dots, x_i)x_n \end{bmatrix},$$

renders control system Σ δ_\exists -GAS.

Proof. The coordinate transformation $\eta = \phi(\xi)$ transforms the control system $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ with f of the form (3.18) to the control system $\Sigma' = (\mathbb{R}^n, \mathcal{U}, f')$, where $f' = \phi_* f$. It can be easily checked that f' has the following form:

$$(3.24) \quad \begin{aligned} f'_1(y, u) &= h'_1(y_1) + y_2, \\ f'_2(y, u) &= h'_2(y_1, y_2) + y_3, \\ &\vdots \\ f'_{n-1}(y, u) &= h'_{n-1}(y_1, \dots, y_{n-1}) + y_n, \\ f'_n(y, u) &= h'_n(y) + g'(y)u, \end{aligned}$$

where $h'_i : \mathbb{R}^i \rightarrow \mathbb{R}$, for $i = 1, \dots, n$, are smooth functions, $g' = \prod_{i=1}^n g_i$, and $y \in \mathbb{R}^n$ is the state of Σ' . As proved in Theorem 3.1, the state feedback control law k , defined in (3.2), makes the metric G_n , defined in (3.13), a contraction metric, with respect to states, for the control system Σ' and with the contraction rate λ . As proved in Lemma 3.7, the metric $\phi^* G_n$ is a contraction metric, with respect to states, with the contraction rate λ , for the control system Σ , equipped with the state feedback control law $k(\phi(x), \hat{u})$. Since $(\mathbb{R}^n, \mathbf{d}_{G_n})$ is a complete metric space and ϕ is an isometry, $(\mathbb{R}^n, \mathbf{d}_{\phi^* G_n})$ is also a complete metric space [Lee03]. Therefore, the state feedback control law $k(\phi(x), \hat{u})$ makes the control system Σ δ_\exists -GAS. \square

The δ_3 -ISS version of Theorem 3.9 is given by the following result.

Theorem 3.10. *Let $\Sigma = (\mathbb{R}^n, \mathcal{U}, \mathcal{U}, f)$ be a control system where f is of the form (3.18). The state feedback control law $u = k(\phi(x), \hat{u})$, where k and ϕ were defined in (3.14) and (3.23), respectively, renders control system Σ δ_3 -ISS with respect to the input \hat{u} .*

Proof. By following the same steps as in the proof of Theorem 3.9, and using Lemma 3.8, we obtain that the state feedback control law $u = k(\phi(x), \hat{u})$ makes the metric ϕ^*G_n a contraction metric, with respect to states and inputs, for Σ and with the contraction rate λ . Hence, the control system Σ , equipped with the state feedback control law $k(\phi(x), \hat{u})$, is δ_3 -ISS with respect to the input \hat{u} . \square

Remark 3.11. *Although we only discussed single input control systems, extensions to multi input control systems are straightforward using the techniques in [KKK95].*

4. BACKSTEPPING CONTROLLER DESIGN FOR A SYNCHRONOUS GENERATOR

We illustrate the results in this paper on a synchronous generator [RGHS01] connected through a transmission line to an infinite bus. The control system $\Sigma = (\mathbb{R}^3, \mathcal{U}, \mathcal{U}, f)$ with f of the form:

$$(4.1) \quad \begin{aligned} f_1(x, u) &= x_2, \\ f_2(x, u) &= -Ex_2 + FP_{m0} + V_s Ge_{q0} \sin(\delta_0 + x_1) + V_s G \sin(\delta_0 + x_1)x_3, \\ f_3(x, u) &= -Ix_3 + JV_s \sin(\delta_0 + x_1)x_2 - Ie_{q0} + IK_c u, \end{aligned}$$

models a synchronous generator connected to an infinite bus. In the aforementioned model, x_1 is the deviation of the power angle, x_2 is the relative speed of the rotor of the generator, x_3 is the deviation of the quadrature axis voltage of the generator, δ_0 is the operating point of the power angle, P_{m0} is the operating point of the mechanical input power, e_{q0} is the operating point of the quadrature axis voltage of the generator, V_s is the infinite bus voltage, K_c is the gain of the excitation amplifier, and u is the input of the silicon-controlled rectifier amplifier of the generator. Other parameters in (4.1) are given by: $E = \frac{D}{2H}$, $I = \frac{1}{T'}$, $F = \frac{\omega_0}{2H}$, $G = -\frac{\omega_0}{2H} \frac{1}{X_{qs}}$, and $J = \frac{X_q - X'_d}{X'_{ds}}$, where D is the per-unit damping constant, H is the inertia constant, ω_0 is the synchronous generator speed, $T' = \frac{X'_{ds}}{X_{qs}} T'_{d0}$, $X_{qs} = X_T + \frac{1}{2}X_L + X_q$, $X'_{ds} = X_T + \frac{1}{2}X_L + X'_d$, T'_{d0} is the direct axis transient short-circuit time constant, X_T is the reactance of the transformer, X_q is the quadrature axis reactance, X'_d is the direct axis transient reactance and X_L is the reactance of the transmission line. We assume that $\sin(\delta_0 + x_1)$ is nonzero over the domain of the interest.

The control system (4.1) is of the form (3.18). The coordinate transformation (3.23), given by:

$$(4.2) \quad \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \phi(\xi) = \begin{bmatrix} \xi_1 \\ \xi_2 \\ V_s G \sin(\delta_0 + \xi_1)\xi_3 \end{bmatrix},$$

transforms the control system $\Sigma = (\mathbb{R}^3, \mathcal{U}, \mathcal{U}, f)$ to the control system $\Sigma' = (\mathbb{R}^3, \mathcal{U}, \mathcal{U}, f')$ with $f' = \phi_* f$ of the form:

$$(4.3) \quad \begin{aligned} f'_1(y, u) &= h'_1(y_1) + y_2 = y_2, \\ f'_2(y, u) &= h'_2(y_1, y_2) + y_3 = -Ey_2 + FP_{m0} + V_s Ge_{q0} \sin(\delta_0 + y_1) + y_3, \\ f'_3(y, u) &= h'_3(y) + g'(y)u = -IV_s Ge_{q0} \sin(\delta_0 + y_1) + JV_s^2 G \sin^2(\delta_0 + y_1)y_2 - Iy_3 + \\ &\quad \cot(\delta_0 + y_1)y_2y_3 + IK_c V_s G \sin(\delta_0 + y_1)u. \end{aligned}$$

By using the results in Theorem 3.4 for a control system of the form (4.3) and for $\lambda = 2$, we have:

$$\begin{aligned}\phi_1(\eta_1) &= -\eta_1, \\ \phi_2(\eta_1, \eta_2) &= -2\eta_1 + (E - 2)\eta_2 - FP_{m0} - V_s Ge_{q0} \sin(\delta_0 + \eta_1), \\ k_3(\eta) &= (-5 + 3E - E^2)\eta_2 - 3\eta_1 + (E - 3)\eta_3 + (E - 3)FP_{m0} \\ &\quad + (E - 3)V_s Ge_{q0} \sin(\delta_0 + \eta_1) - V_s Ge_{q0} \cos(\delta_0 + \eta_1)\eta_2.\end{aligned}$$

Therefore, the state feedback control law:

$$\begin{aligned}(4.4) \quad k(\eta, \hat{v}) &= \frac{1}{g'(\eta)} [k_3(\eta) - h'_3(\eta)] + \frac{1}{g'(\eta)} \hat{v} \\ &= \frac{1}{IK_c V_s G \sin(\delta_0 + \eta_1)} \left[(-5 + 3E - E^2)\eta_2 - 3\eta_1 + (E - 3 + I)\eta_3 + (E - 3)FP_{m0} + \right. \\ &\quad \left. (E - 3 + I)V_s Ge_{q0} \sin(\delta_0 + \eta_1) - V_s Ge_{q0} \cos(\delta_0 + \eta_1)\eta_2 - JV_s^2 G \sin^2(\delta_0 + \eta_1)\eta_2 - \right. \\ &\quad \left. \cot(\delta_0 + \eta_1)\eta_2\eta_3 \right] + \frac{\hat{v}}{IK_c V_s G \sin(\delta_0 + \eta_1)},\end{aligned}$$

makes the control system Σ' δ_3 -ISS with respect to the input \hat{v} . The corresponding contraction metric, with respect to states and inputs, for the control system (4.3) is given by:

$$G(y) = \begin{bmatrix} \begin{bmatrix} 1 + \left(\frac{\partial \phi_1}{\partial y_1}\right)^T \frac{\partial \phi_1}{\partial y_1} & -\left(\frac{\partial \phi_1}{\partial y_1}\right)^T \\ -\frac{\partial \phi_1}{\partial y_1} & 1 \end{bmatrix} & \begin{bmatrix} \left(\frac{\partial \phi_2}{\partial z_2}\right)^T \frac{\partial \phi_2}{\partial z_2} & -\left(\frac{\partial \phi_2}{\partial z_2}\right)^T \\ -\frac{\partial \phi_2}{\partial z_2} & 1 \end{bmatrix} \\ \begin{bmatrix} 2 + (2 + V_s Ge_{q0} \cos(\delta_0 + y_1))^2 & -2E + 5 - (E - 2)V_s Ge_{q0} \cos(\delta_0 + y_1) & 2 + V_s Ge_{q0} \cos(\delta_0 + y_1) \\ -2E + 5 - (E - 2)V_s Ge_{q0} \cos(\delta_0 + y_1) & (E - 2)^2 + 1 & 2 - E \\ 2 + V_s Ge_{q0} \cos(\delta_0 + y_1) & 2 - E & 1 \end{bmatrix} \end{bmatrix},$$

where $z_2^T = [y_1 \ y_2]^T$. By using Theorem 3.10, the state feedback control law (4.4), and the coordinate transformation (4.2), we obtain the state feedback control law $k(\phi(x), \hat{u})$ making Σ δ_3 -ISS with respect to the input \hat{v} . The corresponding contraction metric, with respect to states and inputs, for the control system Σ is given by:

$$(\phi^* G)(x) = \Theta^T(x) G(\phi(x)) \Theta(x),$$

where

$$(4.5) \quad \Theta(x) = \frac{\partial \phi}{\partial x}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ V_s G \cos(\delta_0 + x_1)x_3 & 0 & V_s G \sin(\delta_0 + x_1) \end{bmatrix}.$$

Since the map ϕ does not transform the first and the second coordinates and the metric G is only a function of the first coordinate, we have $G(\phi(x)) = G(x)$.

5. DISCUSSION

In this paper we developed a backstepping procedure to design controllers enforcing incremental stability. Where before we could apply backstepping to construct stabilizing controllers, we can now apply the results in this paper to construct incrementally stabilizing controllers.

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